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Normal Forms for Representations of Representation-finite Algebras

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Using the CREP system we show that matrix representations of representation-finite algebras can be transformed into normal forms consisting of $(0, 1)$-matrices.

1. Introduction and Main Result

Why is representation theory of algebras important for applications? Besides well-known answers like the occurrence of structures like symmetries, operator algebras or quantum groups there is a less popular and more naive answer to this question. Namely, representation theory of finite-dimensional algebras deals with normal forms of matrices. But finding normal forms for matrices is an important technique in many areas where mathematics is applied. A student probably encounters this for the first time when trying to solve a system of linear differential equations describing some physical or biological system or when trying to find the main axes of a symmetric matrix describing the inertia or the magnetic momentum of a solid body.

Representation theory of finite-dimensional algebras provides a general environment to deal with problems of this shape. In our paper we will show how a symbolic computation is applied to obtain information about certain types of normal forms. The symbols we use are finite partially ordered sets and finite quivers. Thus they are of combinatorial nature and, therefore, well-suited for computer implementation.

In fact, our aim is to use the computer classification of the exceptional sincere simply connected algebras (see Dräxler, 1989, 1990) to show that each indecomposable representation of a representation-finite algebra has a normal form whose matrices only have the entries 0 and 1. Note, that a similar result is proved in Ringel for normal forms of exceptional representations of hereditary algebras.

Let us explain how the computation is performed in practice. The essential step in the proof of our result is a case by case inspection in a database containing the list of all exceptional sincere simply connected algebras. This database and the algorithms needed for the inspection are provided by the program system CREP (Combinatorial REPresentation theory) developed in Bielefeld and available from the ftp-server `ftp.uni-bielefeld.de` in the directory `pub/math/f-d-alg`. In this directory one will also find the manual Dräxler and Nörenberg. CREP behaves like a collection of packages of procedures for the computer algebra system Maple. The required database is contained in a package called `esrdalg`.

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We want to finish the introduction by briefly describing the structure of the paper. In Section 2 we will provide the necessary background from the representation theory of quivers and algebras. In Section 3 we will state and prove the main result. Finally, in Section 4 we will make some final remarks on possible further developments.

2. Normal Forms for Representations of Quivers and Algebras

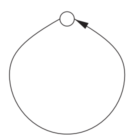
Let us explain how representation theory of finite dimensional algebras deals with normal forms of matrices. For this purpose we need concepts like quivers and their representations introduced in Gabriel (1972). Recall that a *quiver* Q is nothing but a directed graph with a set of vertices Q_0 and a set of arrows Q_1 such that each arrow α in Q_1 has a unique initial point $s(\alpha)$ and a unique final point $t(\alpha)$ in Q_0. For a given field k the *path algebra* kQ is the vector space having as basis the paths in the quiver Q. A path of positive length p in Q is a sequence $\alpha_p \dots \alpha_1$ of arrows such that $s(\alpha_q) = t(\alpha_{q-1})$ for all $q = 2, \dots, p$. Moreover, to each vertex i in Q_0 there is attached a path 1_i of length 0 starting and ending in i. The path algebra kQ is endowed with the multiplication which is defined for two paths v and u as the concatenation of the two paths if the final point of u coincides with the initial point of v and 0 otherwise. Note, that kQ is not finite-dimensional if Q has an oriented cycle. Representations of a path algebra kQ are easy to describe as representations of Q (over the given field k). A *representation* X of Q is given by a vector space $X(i)$ for each vertex i in Q_0 and a linear map $X(\alpha) : X(s(\alpha)) \to X(t(\alpha))$ for each arrow α in Q_1.

A *homomorphism* from a representation X to a representation Y of a quiver Q is a family $\phi = (\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : X(i) \to Y(i)$ such that $Y(\alpha)\phi_{s(\alpha)} = \phi_{t(\alpha)}X(\alpha)$ for all arrows α. Such a homomorphism is called an isomorphism if all the maps ϕ_i are bijective. Two representations X, Y are said to be isomorphic if there exists an isomorphism from X to Y.

A representation X of Q is called *finite-dimensional* if all vector spaces $X(i)$ are finite-dimensional say of dimension $n(i)$. By choosing bases, each space $X(i)$ can be identified with $k^{n(i)}$. This forces each map $X(\alpha)$ to be a (left) multiplication by a $n(t(\alpha)) \times n(s(\alpha))$-matrix over k which we also denote by $X(\alpha)$. A representative of an isomorphism class of finite-dimensional representations having this shape is called a *normal form*.

The direct sum of a collection $X_1, \dots, X_m$ of representations of a quiver Q is defined in the obvious way by building the direct sum of the vector spaces $X_1(i), \dots, X_m(i)$ for each vertex i in Q_0. A representation X of Q is said to be indecomposable if it is not isomorphic to the direct sum of two non-zero representations. By the theorem of Krull–Remak–Schmidt any finite-dimensional representation is isomorphic to a direct sum of finitely many indecomposable representations which are unique up to permutation and isomorphism. Thus it is sufficient to find normal forms for all isomorphism classes of indecomposable representations.

The most popular normal forms are the Jordan block matrices which are the normal forms for the indecomposable representations of the one loop quiver.



Another classical example is the normal form for matrix pencils (see Gantmacher, 1966) which is the normal form for the indecomposable representations of the Kronecker quiver below. For an application in perturbation theory we refer to Aronszajn and Fixman (1968).

$$\circ \rightrightarrows \circ$$

To what extent are quivers and their path algebras typical in the representation theory of algebras? If we wish to study a finite-dimensional representation X of a given k-algebra A, then we clearly can consider X as representation of the factor algebra of A by the annihilator ideal of X. Since this factor algebra is finite-dimensional, it suffices to consider finite-dimensional algebras when looking for normal forms for finite-dimensional indecomposable representations. It was emphasized by Gabriel (see Gabriel, 1972, 1980) that any finite-dimensional algebra over an algebraically closed field k (like the complex numbers $\mathbb{C}$) can be considered (up to Morita equivalence) as a factor algebra kQ/I of the path algebra kQ associated with a finite quiver Q. The ideal I of kQ can be chosen to be admissible which means that it is contained in the square of the ideal of kQ generated by the arrows but contains some power of this ideal. Note, that such an ideal can always be generated by finitely many generators $\rho_1, \ldots, \rho_r$ such that each of these 'relations' ρ_l is a linear combination $\rho_l = \sum_{q=1}^{m} \lambda_q u_q$ where $\lambda_q \in k$ and all the u_q are paths from a vertex i to a vertex j of Q_0. We observed earlier that representations of a quiver algebra kQ can be identified with the representations of Q over k. This transfers easily to representations of kQ/I. Namely, the representations of this factor algebra are just those representations X of Q satisfying all the relations ρ_l meaning that $0 = \sum_{q=1}^{m} \lambda_q X(u_q)$. Here we use the abbreviation $X(u) := X(\alpha_p) \cdots X(\alpha_1)$ when considering a path $u = \alpha_p \ldots \alpha_1$ in Q. Thus classifying the indecomposable representations of a finite-dimensional algebra $A = kQ/I$ amounts to finding normal forms for the indecomposable representations of Q satisfying the relations $\rho_1, \ldots, \rho_r$ which generate the ideal I.

In the last 30 years much theoretical effort was made to understand the representations of a finite-dimensional algebra. But due to the combinatorial nature of quivers concrete algorithms were also developed and implemented to classify certain classes of algebras. However, the normal forms of indecomposable representations of a given algebra will usually not only depend on discrete parameters. An exception is furnished by the *representation-finite* algebras defined by the property that there are only finitely many isomorphism classes of indecomposable representations or, in other words, the list of normal forms of indecomposable representations is finite. There are many results demonstrating representation-finite algebras culminating in the theorem on the existence of multiplicative bases proved in Bautista *et al.* (1985). In Bautista *et al.* (1985) to each representation-finite finite-dimensional algebra A a so-called *standard form* kQ/I is associated which is also representation-finite. The ideal I is especially nice having generators consisting of paths and differences of paths. Of particular interest for us is that the standard form is isomorphic to A (up to Morita equivalence) provided that the characteristic of k is different from 2. Moreover, if one is only interested in one particular indecomposable representation, then by passing to the annihilator factor algebra one may assume that the representation is faithful. In this case by Bongartz (1985) the algebra A is isomorphic to its standard form also if the field k is of characteristic 2.

Not so much is known about normal forms for indecomposable representations of representation-finite algebras. In this paper we want to improve this situation by presenting the following result for standard forms.

3. The Main Result

THEOREM 3.1. *Let $A = kQ/I$ be the standard form of a representation-finite algebra over an algebraically closed field k. Then for each isomorphism class of indecomposable representations of kQ/I there is a normal form X such that for all arrows α in the quiver Q the matrix $X(\alpha)$ has only the entries 0 and 1.*

PROOF. The proof will be divided into several steps.

Step 1: From Bautista *et al.* (1985) we know that for the standard form kQ/I there is a universal Galois covering [see Gabriel (1981) for the definition and properties of the needed coverings] such that each finite convex subcategory is simply connected. But simply connected algebras as considered in Bautista *et al.* (1985) are *completely separating* algebras in the sense of Dräxler (1994a). Moreover, the push-down functor associated with the universal covering uses the matrices attached to the arrows of the covering as blocks for the matrices attached to the arrows of Q. Therefore, in order to prove our theorem, we may assume that $A = kQ/I$ itself is completely separating.

Step 2: A completely separating algebra can be presented in a way which involves only combinatorial data. Namely, by Dräxler (1994a) there is a finite partially ordered set S such that $A = kS/J$ where J is an admissible ideal of the incidence algebra kS of S over k. Recall that the pairs (y, x) of elements of S satisfying $x \geq y$ form a k-basis of kS. The multiplication is defined on the base vectors by $(z, y)(y, x) := (z, x)$ and $(z, y')(y, x) := 0$ for $y \neq y'$. That the ideal J is admissible means in this case that J is generated by elements $\rho = (y, x)$ such that $x \neq y$ are not direct neighbours in S. Thus, to describe $A = kS/J$ one has to store the finite partial orders set S (say by its Hasse diagram) and the finite list of pairs $\rho = (y, x)$ of generators of J.

Representations of kS can be identified with representations of S over k. Such a representation X is given by a vector space $X(x)$ for each element x of S and a linear map $X(y, x) : X(x) \to X(y)$ for each $y \leq x$. In addition, these maps have to satisfy $X(z, x) = X(z, y)X(y, x)$ for all $z \leq y \leq x$. Representations of the factor algebra kS/J are representations X of S where $X(y, x) = 0$ for every generator $\rho = (y, x)$ of J.

How are Q and S related for a completely separating algebra $kS/J = A = kQ/I$? The quiver Q is nothing but the usual Hasse diagram of S which means that the vertices of Q are the elements of S and there is an arrow from x to y in Q provided that $y < x$ are direct neighbours in S (i.e. there is no element z of S such that $y < z < x$). Actually, the map sending an arrow to its corresponding base element (y, x) yields the required identification of kQ/I and kS/J. Therefore, in order to prove our theorem it is sufficient to show the following: for each isomorphism class of indecomposable finite-dimensional representations of kS/J there exists a representative Y such that $Y(y, x)$ is given by a matrix whose coefficients are only 0 or 1 for all direct neighbours $y < x$ in S. This will be done by constructing suitable bases in the spaces $X(x)$ for an arbitrarily chosen representative X.

Step 3: Let us first introduce a class of representations of kS/J which by definition has a basis of the shape we desire. A subset T of S is said to be *strictly convex* if it is a convex connected subset of S which does not contain y and x simultaneously for any generator $\rho = (y, x)$ of J. It is rather obvious that from a strictly convex subset T one gets an indecomposable representation X_T of kS/J by putting $X_T(x) := k$ for $x \in T$, $X_T(x) := 0$ for $x \notin T$, $X_T(y, x) := 1$ for $x, y \in T$ and $X_T(y, x) := 0$ in all other cases.

For computational purposes it is crucial, that it is easy to check if a given indecomposable representation is of the shape X_T. Namely, it is sufficient to know that X is *thin* (see Dräxler, 1994a). This means that $X(x)$ is at most one-dimensional for all x in S. An example for a strictly convex subset of S is the set $\check{s}$ which for a given element s of S is the set of all elements $y \leq s$ in S such that (y, s) is not contained in J. Note, that $P_s := X_{\check{s}}$ is the indecomposable projective representation of kS/J attached to s.

Step 4: Unfortunately, not every indecomposable representation of kS/J is isomorphic to some X_T. On the other hand, it was mentioned in Dräxler (1990) that, using a case by case inspection on a computer, one can see that each indecomposable representation X is a fibre sum over thin start modules with respect to a suitable element s of S. Recall, that an indecomposable module Y is called a *start module* with respect to s provided $Y(s) \neq 0$ but $\tau Y(s) = 0$ where τ is the Auslander–Reiten translation (see Auslander *et al.*, 1995). The presentation of an indecomposable representation X as fibre sum over thin start modules is the essential tool in our proof which is obtained by a computer calculation. We will outline how this is obtained using the CREP system.

By restricting to the support algebra of X, we may assume that X is *sincere*, i.e. $X(x) \neq 0$ for all x in S. An algebra having a sincere indecomposable representation is called *sincere* as well. The sincere representation-finite completely separating algebras come in 24 regular families (see Bongartz, 1982) and 16 344 exceptional cases (see Dräxler, 1989, 1990). The list of these exceptional cases is available via the package `esrdalg` of CREP. For all algebras in this data base the Auslander–Reiten quivers can be calculated using the procedure `preproj`. For details about Auslander–Reiten quivers we refer to Auslander *et al.* (1995) or Gabriel and Roiter (1992). For us it is only important to remark that the Auslander–Reiten quivers of the algebras kS/J in the data base are finite directed quivers which can be handled easily by the system. To all occurring indecomposable sincere representations X the computer will find an element s of S such that all predecessors of X in the Auslander–Reiten quiver of kS/J which are start modules for s happen to be thin. This is the result we need. So far we have only dealt with the exceptional algebras. But for the algebras in the regular families classified in Bongartz (1982) this is immediate to see.

Step 5: For details about fibre sum functors we refer to Dräxler (1990) and Dräxler (1994b). We only need two properties. The first one is that we may write X as a fibre sum with respect to an arbitrary $s \in S$. This means the existence of an exact sequence

$$P_s^m \xrightarrow{\delta} \oplus_{i=1}^n Y_i \xrightarrow{\sigma} X \to 0$$

where all the modules Y_i are start modules with respect to s. Thus all the start modules Y_i have to be predecessors of X in the Auslander–Reiten quiver of kS/J. Choosing s as before we therefore can assume that each module Y_i is thin, hence of the shape $Y_i = X_{T_i}$, for some strictly convex subset T_i of S containing s. The second property we need is, that the map δ can be considered as an indecomposable matrix representation (see Nazarova and Roiter, 1975) of the set $\{T_1, \ldots, T_n\}$ which is partially ordered by putting $T_i \geq T_j$ if there is a homomorphism $\phi : X_{T_i} \to X_{T_j}$ such that $\phi_s \neq 0$. Let us put $Y := \oplus_{i=1}^n X_{T_i}$. If we equip the spaces $P_s^m(x)$ resp. $Y(x)$ with the canonical bases induced from the summands P_s resp. X_{T_i}, then the linear map $\delta_s : k^m \to k^n$ is given by one of Kleiner's exact matrix representations of partially ordered sets of finite type displayed in Kleiner (1975).

Step 6: Now the crucial observation is that all matrices in Kleiner's list only have the coefficients 0 and 1. Moreover, it is easy to see that the cokernel map $\sigma_s : k^n \to X(s)$ can also be taken from Kleiner's list after choosing appropriate bases in $X(s)$ and possibly replacing δ by $\psi\delta$ using some automorphism ψ of Y. This is the information we will need to complete our proof.

A $m \times n$-matrix E over the field k is said to be elementary if the non-zero columns form a family of pairwise different canonical base vectors. Examples of elementary matrices are unit matrices. An elementary matrix E is said to be a projection if the rank of E is m and an injection if the rank of E is n. We will need that for a matrix C having only the entries 0 and 1, the products EC and $E'C$ with elementary matrices E and E' also have only the entries 0 and 1. From the definition of X_T we obtain that all $Y(y,x)$ are elementary matrices.

Let us now consider the elements y of S such that $y < s$ and $X(y) \neq 0$. Since $s \in T_i$ we obtain that $Y(y,s)$ is an epimorphism given by a projection matrix. Due to the following commutative diagram $X(y,s)$ also has to be an epimorphism.

$$\begin{tikzcd} Y(s) \arrow[r, "\sigma_s"] \arrow[d, "Y(y,s)"'] & X(s) \arrow[d, "X(y,s)"] \\ Y(y) \arrow[r, "\sigma_y"] & X(y) \end{tikzcd}$$

Consequently, using $X(y,s) = X(y,x)X(x,s)$, we can choose bases simultaneously in all $X(y)$ with $y < s$ such that $X(y,x)$ is represented by a projection matrix for all $y < x \leq s$.

Let us look at the commutative diagram once more. As $Y(y,s)$ is given by a projection matrix, there is an injection matrix L such that $Y(y,s)L$ is the identity matrix. Hence, σ_y can be calculated as $X(y,s)\sigma_s L$. Because $X(y,s)$ and L are elementary matrices, σ_y is also given by a matrix which has only the entries 0 and 1.

If $y \leq x$ are elements of S such that $x \not\leq s$ and $y \not\leq s$, then we can choose σ_x and σ_y to be identities and $X(y,x) = Y(y,x)$ is represented by an elementary matrix.

It remains to deal with the case that x is an upper neighbour of y in S and $x \not\leq s$, $y \leq s$. We obtain the following commutative diagram:

$$\begin{tikzcd} Y(x) \arrow[r, equal] \arrow[d, "Y(y,x)"'] & X(x) \arrow[d, "X(y,x)"] \\ Y(y) \arrow[r, "\sigma_y"] & X(y) \end{tikzcd}$$

With respect to the chosen bases $Y(y,x)$ is given by an elementary matrix and σ_y by a matrix which has only the entries 0 and 1. Hence, the matrix $X(y,x)$ also has this property and our proof is complete. □

4. Final Remarks

We used a computer calculation to show the existence of normal forms with entries 0 and 1 only. With some more computational effort we would be able to calculate the normal forms explicitly for the representation-finite algebras which are simply connected. Nevertheless, for the general case one would still need implementations of algorithms producing the standard form of a representation-finite algebra and its universal Galois covering. This already exists in CREP for the special case of monomial algebras. We refer to Nörenberg (1998) for details.

Presently, in the representation theory of finite-dimensional algebras, the tame algebras are a most interesting topic. For them, the indecomposable representations of fixed dimension lie up to isomorphism on finitely many one-parameter families. One would need the corresponding data bases and algorithms to obtain similar results on normal forms for representations of tame algebras as we did for representation-finite algebras.

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